On certain zeta integrals
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Integral representations of $L$-functions

**Common features:**

- Use tractable integrals involving automorphic forms + *other stuff* to study $L$-functions, etc.
- Adélic formulation.
- Meromorphic continuation, bounded in vertical strips, etc.
- Functional equation(s).
- Euler product; theory of local integrals.

Usually called *zeta integrals* — the core of the $L$-function machine.
**Goal**: study the standard $L$-functions of $GL_n$, both locally and globally. We begin with the case over a local field $F$

- $GL_n \times GL_n$-equivariant embedding $GL_n \hookrightarrow Mat_n$.
- $\xi \in S(Mat_{n\times n})$: Schwartz-Bruhat functions.
- For an irrep $\pi$ of $GL_n(F)$, integrate against $\xi$ the matrix coefficients $\langle \tilde{v}, \pi(\cdot)v \rangle$ (= images under equivariant $\pi \boxtimes \tilde{\pi} \to C^\infty(GL_n)$, satisfying multiplicity one):

$$Z(s, \pi, v \otimes \tilde{v}, \xi) := \int_{GL(n,F)} \xi(x)\langle \tilde{v}, \pi(x)v \rangle |\det(x)|^{s+\frac{n-1}{2}} \ d^\times x.$$ 

Facts: convergence for $\text{Re}(s) \gg 0$, meromorphic/rational continuation.
**Local functional equation:** there exists a meromorphic/rational function $\gamma(s, \pi)$ such that

$$Z(1 - s, \tilde{\pi}, \tilde{\nu} \otimes \nu, \mathcal{F}\xi) = \gamma(s, \pi)Z(s, \pi, \nu \otimes \tilde{\nu}, \xi)$$

where $\mathcal{F}$ is the Fourier transform on $\text{Mat}_{n \times n}(F)$ relative to $(X, Y) \mapsto \psi(\text{tr}XY)$, for a chosen additive character $\psi$.

**Relation to $L$-functions:** taking greatest common divisor

$$Z(s, \pi, \xi) \sim L(s, \pi).$$

**Shift in $s$:** the $-\frac{1}{2}$ is nice, but where does $n/2$ come from?

Note: $|\det|^{n}d^{\times}x = dx$
Global case: let $A = A_F$, $\pi$ be cuspidal automorphic, $\phi \in \pi$ and $\tilde{\phi} \in \tilde{\pi}$ be cusp forms,

$$
\int_{GL(n,A)} \beta(x) \xi(x) |det(x)|^{s+\frac{n-1}{2}} d^x x
$$

where $\Re(s) \gg 0$ and

- $x\phi(\cdot) = \phi(\cdot x)$, $\beta(x) = \langle \tilde{\phi}, x\phi \rangle_{\text{Pet}}$ (the Petersson pairing) is the global matrix coefficient: it factorizes,
- $\xi \in \mathcal{S}(\text{Mat}_{n \times n}(A))$.

It serves as a prototype for many other integral representations of $L$-functions.

Exercise: rewrite it as a convergent integral involving $\phi$ and $\tilde{\phi}$ on $\text{diag}(a) \text{GL}(n, F')^2 \backslash \text{GL}(n, A)^2$, for a suitable central connected subgroup $a \subset \text{GL}(n, F_{\infty})$. 
A typical set-up: let $W$ be a symplectic $F$-vector space, $\dim W \geq 2$ and

$W^\square := (W, \langle , \rangle) \oplus (W, -\langle , \rangle)$.

- $G = \text{Sp}(W)$, $H = \text{Sp}(W^\square)$;
- $P \subset H$ is the stabilizer of the Lagrangian $x_0 := \text{diag}(W)$ in $W^\square$, $P_{ab} \simeq \mathbb{G}_m$, and we have a natural algebraic character $\det_P : P_{ab} \to \mathbb{G}_m$;
- $G \times G \hookrightarrow H$ naturally;
- since $(G \times G) \cap P = \text{diag}(G)$, one obtains a $G \times G$-equivariant embedding $G \hookrightarrow P \backslash H$ with open dense image;
- the boundaries are negligible: every $\gamma \in \partial H$ is stabilized by the unipotent radical of some proper parabolic. Morally, this means $\partial H$ does not interact with cusp forms.
Global integrals. Let $F$ be a number field. Let $\sigma \boxtimes \pi \boxtimes \tilde{\pi}$ be a cuspidal automorphic representation of $(P_{ab} \times G \times G)(\mathbb{A}_F)$. Form

$$
\int_{[G \times G]} E_f(\sigma, s)(x, x') \phi(x) \phi'(x') \, dx \, dx'
$$

- $\phi \in \pi$, $\phi' \in \tilde{\pi}$;
- $f$ a “good section” for $I^H_P(\sigma \otimes \det_P^s)$ (unitary parabolic induction), parametrized by $s \in \mathbb{C}$;
- $E_f(\sigma, s) : H(F) \backslash H(\mathbb{A}_F) \to \mathbb{C}$ the Eisenstein series made from $f$.

Properties of Eisenstein series (eg. intertwining operators) $\implies$ meromorphic continuation, functional equation...
Local integrals. Let $\sigma \boxtimes \pi \boxtimes \tilde{\pi}$ be a smooth irreducible representation of $(P_{ab} \times G \times G)(F)$. Form

$$\int_{G(F)} f(x_0(x, 1)) c_\pi(x) \, dx$$

where

- $f : R_u(P) \backslash H(F) \to \mathbb{C}$ is a “good section” for $I_P^H(\sigma \otimes \det_P^s)$ as before;
- $c_\pi \in C^\infty(G(F))$ is a matrix coefficient of $\pi$.

By taking gcd, these integrals represents the Rankin-Selberg $L$-function $L(s + \frac{1}{2}, \chi \otimes \pi)$.

Furthermore, the global integral factorizes into local ones.
The doubling construction can be adapted to $G = \mathrm{GL}(n)$, $H = \mathrm{GL}(2n)$ to yield the Godement-Jacquet integrals. Piatetski-Shapiro and Rallis claimed that the doubling method is the correct generalization of Godement-Jacquet theory.

Things can go the other way around. Braverman-Kazhdan (2002)

Use the affine embedding $P_{ab} \times G \hookrightarrow P_{\text{der}} \backslash H =: X$. Replace good section $f$ by suitable test function on $X(F)$. Normalized intertwining operators for $I^P_H(\cdots)$ get replaced by “Fourier transforms”. The resulting theory resembles the original Godement-Jacquet.

Furthermore: in the unramified setting, there is a distinguished $P_{ab} \times H(\mathfrak{d}_F)$-invariant test function $\xi^\circ$ (denoted by $c_{P,0}$ in loc. cit.) whose values are closely related to the trace-of-Frobenius of the IC sheaf on Drinfeld’s compactification $\overline{\text{Bun}_P}$ of $\text{Bun}_P$, the moduli stack of $P$-bundles over a smooth projective $\mathbb{F}_q$-curve. Why?
Igusa zeta integrals

One of the interesting integrals in the pre-Langlands era. It originates from ideas of Gelfand et al. on complex powers. Let $F$ be a local field. We want to study integrals

$$\int_{X(F)} |f|^s \xi, \quad \text{Re}(s) \gg 0$$

for

- appropriate varieties $X$ (often: affine $n$-space),
- $\xi \in C_c^\infty(X(F))$,
- $f \in F[X]$: “interesting” function.

Seek for meromorphic continuation in $s$, location of poles, evaluation at special $\xi$, etc.

It is even more interesting if symmetries enter into this picture, by letting a group $G$ act on $X$ so that $f$ is an eigenfunction.
Prehomogeneous zeta integrals

Let $X$ be an $F$-vector space, on which a reductive group $G$ acts with dense open orbit $X^+$. Assume that

1. $\partial X = X \setminus X^+$ is a hypersurface ($f = 0$); we assume $f \in F[X]$ transforms with eigencharacter $\omega : G \to \mathbb{G}_m$;
2. $X^+(F)$ is a single $G(F)$-orbit (for simplicity).

Define

$$Z(s, \xi) := \int_{X(F)} |f|^s \xi, \quad \xi \in \mathcal{S}(X).$$

Local functional equation (M. Sato, T. Shintani, F. Sato, et al)

Assume $X$ “regular” (hence $\tilde{X}$ is prehomogeneous) and $X^+(F)$ is a single $G(F)$-orbit. $Z(\ast - s, \mathcal{F}(\cdot))$ and $Z(s, \cdot)$ can be related up to “$\gamma$-factor”.

Multiple $G(F)$-orbits in $X^+(F)$ leads to functional equation with “$\gamma$-matrices”, by considering integrals over each orbit separately.
In comparison with the local Godement-Jacquet and doubling constructions, one may consider

- $\pi$: irrep of $G(F)$,
- $\varphi \in \mathcal{N}_\pi := \text{Hom}_{G(F)}(\pi, C^\infty(X^+))$, and
- study the zeta integral $Z(s, \varphi, \nu, \xi) = \int_{X(F)} \varphi(\nu)|f|^s \xi$.

An attempt in this direction is made by Bopp and Rubenthaler (2005), under various conditions (e.g. assuming $X^+$ is a symmetric space, specific series of $\pi$). One indispensable premise seems to be

$$\dim \mathcal{N}_\pi < \infty.$$
For split reductive $G$, they formulated a vast conjectural generalization of Godement-Jacquet integrals, using

- $G \times G$-equivariant embedding of $G$ into some algebraic monoid $M$
  with unit group $G$ together with $\det_M : M \to \mathbb{G}_a$, restricting to a homomorphism $G \to \mathbb{G}_m$;
- conjectural Schwartz space $S$;
- conjectural Fourier transform, satisfying an even more conjectural Poisson formula.

Conjecturally, integration of matrix coefficients against $\xi \in S$, twisted by $|\det_M|^s$, represents a large family of $L$-functions attached to $\rho : \widehat{G} \to \text{GL}(N, \mathbb{C})$.

$$\rho \leftrightarrow \text{highest weight} \xrightarrow{\text{Vinberg theory}} M = M_\rho$$

Related works: L. Lafforgue, Bouthier-Ngô-Sakellaridis......
Analysis on homogeneous spaces

- $G$ under $G \times G$ is just a special case of homogeneous spaces — the group case.
- A generally accepted framework for doing harmonic analysis: spherical homogeneous spaces, i.e. $\exists$ open Borel orbit.
- Let $X^+$ be spherical homogeneous under $G$ and $\pi$: irrep. Main issues in harmonic analysis include
  - decomposition of $L^2(X^+)$,
  - intertwining operators $\pi \rightarrow C^\infty(X^+)$ yielding the “coefficients” of $\pi$,
  - intertwining operators relating different $X^+$.
- Must also consider spherical embeddings: $G$-equivariant morphisms $X^+ \hookrightarrow X$ with open dense image and normal $X$.

Important works are being done by Sakellaridis-Venkatesh.
Let $H$ be a split reductive group. In the global case, Sakellaridis (2012) proposes to generate many integral representations using

- affine spherical embedding $X^+ \hookrightarrow X$,
- $\forall v$, conjectural Schwartz space $S_v \subset C^\infty(X^+(F_v)))$ from these data.
- for almost all $v \nmid 1$, there should be a distinguished $G(o_v)$-invariant element $\xi_v^0 \in S_v$ (the basic function).

The expected behavior of $\xi \in S_v$ is complicated by the singularities of $X$.

**Example: Bouthier-Ngô-Sakellaridis**

In the Braverman-Kazhdan case $X^+ = G \hookrightarrow X = M_\rho$, suppose everything unramified.

They interpreted $\xi_v^0 \in S_v$ as the trace of Frobenius of an appropriately defined IC sheaf on $L^+X$, the formal arc space attached to $X^+$, by passing to the equal-characteristic case.
It seems that a correct way of viewing zeta integrals is crucial. We try to refine Sakellaridis’ proposal as follows. $G$: split connected reductive group over a local field $F$, char$(F) = 0$, common patterns include

1. an affine $G$-spherical embedding $X^+ \hookrightarrow X$;
2. a Schwartz space $S$ of “test functions” on a $X^+(F)$;
3. integrate the coefficients of an irrep $\pi$ (smooth admissible, SAF if $F \supset \mathbb{R}$) against test functions from $S$;
4. a mechanism to twist coefficients by $|f|^s$ for appropriate $f \in F[X]^G_{\text{eigen}}$;
5. meromorphic/rational continuations of the zeta-integrals/zeta distributions so obtained;
6. functional equation of zeta integrals, which is a manifestation of some “Fourier transform” of Schwartz functions.
Reality check

Any such formalism must give a simple, conceptual explanation of the Godement-Jacquet theory.

Other issues:

- Can we prove anything under such a general framework?
- Clarify the functional-analytic underpinnings.
- Local-global compatibility.
- Try to find manageable examples, for which Schwartz space and Fourier transform are already available.
- It should make zeta integrals appear more natural to an outsider.
Towards a broader framework

\(G: \) split connected reductive group

1. \(X^+ \hookrightarrow X: \) affine spherical embedding.

2. The boundary \(X \smallsetminus X^+\) is the union of prime divisors \(f_i = 0\) (eigencharacter =: \(\omega_i\), \(i = 1, \ldots, r\)).

3. Assume the eigencharacters \(\omega_1, \ldots\) are linearly independent and generate a lattice \(\Lambda\). Write \(|\omega|^\lambda = \prod_i |\omega_i|^{\lambda_i}, \ |f|^\lambda := \prod_i |f_i|^{\lambda_i}\) for \(\lambda = \sum_i \lambda_i \omega_i \in \Lambda \otimes \mathbb{C}\).

4. Assume \(\mathcal{N}_\pi := \text{Hom}_{G(F)}(\pi, C^\infty(X^+))\) is finite-dimensional. Known for \(F = \mathbb{R}\) (Kobayashi-Oshima) or \(F\) non-archimedean and \(X^+\) wavefront (Sakellaridis-Venkatesh). Here we should take continuous Hom.

5. Work with a given Schwartz space \(C^\infty_c(X^+) \subset \mathcal{S} \subset C^\infty(X^+)\), a continuous smooth \(G(F)\)-representation.

\[Z_\lambda(\xi, \varphi(v)) := \int_{X^+} \xi \varphi(v) |f|^\lambda, \quad \varphi \in \mathcal{N}_\pi, \xi \in \mathcal{S}, v \in \pi.\]
Digression: What can we integrate?

- Integration applies to any density, locally written as \(|\omega|\) (\(\omega\): volume form). Density bundle = \(\mathcal{L}\).
- The \(L^2\)-ness makes sense for a \(\frac{1}{2}\)-density, locally as \(|\omega|^{1/2}\).
- Integration of densities is canonical if we fix a Haar measure on \(F\).
- The \(L^2\)-pairing is canonical and invariant under all symmetries of \(X^+(F)\), giving rise to unitary representations.

Hence: \(L^2(X^+)\) stands for the \(L^2\)-sections of the bundle \(\mathcal{L}^{1/2}\) of \(\frac{1}{2}\)-densities. Re-define

- \(S \subset L^2(X^+)\) is \(\mathcal{L}^{1/2}\)-valued,
- \(C^\infty(X^+) := C^\infty(X^+(F), \mathcal{L}^{1/2})\),
- \(\varphi \in N_\pi := \text{Hom}_{G(F)}(\pi, C^\infty(X^+))\).

More generally, we should allow values in some line bundles, eg. the Whittaker-induced case.
What is this good for? Let’s return to the Godement-Jacquet scenario

\[ S := \{ \text{Schwartz-Bruhat half-densities on } \text{Mat}_{n \times n}(F) \} \]

\[ = \{ \xi_0 |dx|^{1/2} : \xi_0 \in S_{\text{usual}} \}, \]

where \( dx \) is a translation-invariant volume form. Therefore

\[ \xi_0 |dx|^{1/2} = \xi_0 |\det x|^{n/2} |\det x|^{-n/2} |dx|^{1/2} = \xi_0 |\det|^{n/2} |d^x x|^{1/2} \]

where \( |d^x x| \) stands for the Haar measure on \( \text{GL}(n, F) \).

- This explains the \( n/2 \)-shift.
- Fix \( \psi \), the Fourier transform for \( S \) is canonically defined and \( \text{GL}(n, F) \)-equivariant (already known to E. Stein \( \leq 1967 \)).

**Lesson:** Don’t trivialize \( L \) even though you can.
Partial results

Assume $F$ non-archimedean and $\forall \xi \in \mathcal{S}$, the support of $\xi$ has compact closure in $X(F)$.

1. Using the asymptotics of coefficients $\varphi(v) \in C^\infty(X^+)$ from Sakellaridis-Venkatesh, one can show the convergence for $\Re(\lambda) \gg 0$ when $X^+$ is wavefront. Idea: use a smooth toroidal compactification of $X^+$ that is compatible with $X^+ \hookrightarrow X$.

2. In the prehomogeneous case with the usual $\mathcal{S}$, Igusa’s theory implies the rationality of zeta integrals.

This framework accommodates:

- the aforementioned generalization of prehomogeneous zeta integrals with $X^+$ spherical and wavefront,
- Godement-Jacquet $\subset$ Braverman-Kazhdan,
- doubling integrals interpreted via the $P_{ab} \times G \times G$-equivariant embedding $X^+ := P_{ab} \times G \hookrightarrow P_{\text{der}}\backslash H \hookrightarrow \overline{P_{\text{der}}\backslash H}^{\text{aff}} = X$. 
In the symplectic case at least, the doubling method can be subsumed into Braverman-Kazhdan. The reason is as follows.

**Theorem (Rittatore)**

View $G$ as a $G \times G$-variety (the “group case”). Its affine spherical embeddings are the same as normal affine algebraic monoids with unit group $G$.

Moreover, the monoid/embedding $X^+ \hookrightarrow X$ in the doubling method for $G = \text{Sp}(2n)$ matches Ngô’s recipe with the expected representation

$$\rho := \text{id} \boxtimes \text{Std} : \mathbb{C}^\times \times \text{SO}(2n + 1, \mathbb{C}) \to \text{GL}(2n + 1, \mathbb{C})$$

of $(P_{ab} \times G)^\wedge = \mathbb{C}^\times \times \text{SO}(2n + 1, \mathbb{C})$. The verification uses

- a closer look at the stratification of $X = P_{\text{der}} \backslash H$ into $P_{ab} \times G \times G$-orbits;
- Luna-Vust classification.
Currently, the formalism does not include Rankin-Selberg integrals on $GL(m) \times GL(n)$ with $n < m$. Doing this will require:
  - allowing homogeneous $G$-spaces that are “Whittaker-induced” from a Levi,
  - understanding the “unfolding” process.

The most accessible case: $X$ smooth. Such $G$-varieties tend to be prehomogeneous (Luna); according to Sakellaridis (2012), the $L$-functions coming from prehomogeneous zeta integrals have been known by other methods.
Relation to spectral decomposition

Generally, every $X^+$ has an abstract Plancherel decomposition

$$L^2(X^+) = \int_{\Pi_{\text{unit}}(G)} \mathcal{H}_\tau \, d\mu(\tau), \quad \mathcal{H}_\tau = \tau \hat{\otimes} \mathcal{M}_\tau.$$ 

In the group case: $X^+ = G$ under $G \times G$-action, Harish-Chandra gives a far-reaching refinement of the decomposition above.

In the Godement-Jacquet case $G = \GL(n, F)$, zeta integrals are related to the spectral decomposition of $L^2(\GL(n, F))$:

1. Plancherel formula describes the image of $\xi \mapsto \pi(\xi) \in \End_C(V_\pi)$, $\xi \in \mathcal{C}_c^\infty(X^+)$, $\pi$ in the tempered dual of $G(F)$;

2. for $\xi \in \mathcal{S}(\Mat_{n \times n})$ and $\Re(\lambda) \gg 0$, the image of $Z_\lambda(\xi, \cdots) \leftrightarrow (\pi \otimes |\det|^\lambda)(\xi)$ has a similar description involving $L(\lambda, \pi)$. 
Lafforgue turned this approach over to study Braverman-Kazhdan program:

Given local factors $L(s, \pi)$, one defines $S$ by spectral means.

This leads the notion of functions of $L$-type, natural formulations of Fourier transforms, etc.

- Heuristically very useful.
- In general: have no $L$-factor at hand, except when $\pi$ is in the principal series, etc.
- Putting the cart before the horse?

**Issue of spectral decomposition**

Clarify the relation between zeta integrals and spectral decomposition of $L^2(X^+)$ for general $X^+$. Deduce this from properties of $S$ (not conversely).
Gelfand-Kostyuchenko method

- \( L^2(X^+) = \int_{\Pi_{\text{unit}}(G)}^{\oplus} \mathcal{H}_\tau \, d\mu(\tau) \), assuming the multiplicity space \( \mathcal{M}_\tau \) is finite-dimensional.

- In general: the decomposition is not defined by operators \( L^2(X^+) \to \mathcal{H}_\tau \) (think of \( L^2(\mathbb{R}) \) — classical Fourier analysis!)

- Rigged Hilbert spaces: \( S \hookrightarrow L^2(X^+) \) dense continuous, requirement: it is pointwise defined by a family \( \alpha_\tau : S \to \mathcal{H}_\tau \) with dense images.

- OK for the minimalist choice \( S = C^\infty_c(X^+) := C^\infty_c(X^+(F), \mathcal{L}^{1/2}) \).

- Sufficient condition for the existence of \( (\alpha_\tau)_\tau : S \) is nuclear separable; this appeared in *Generalized Functions, Vol. IV*.

Can embed

\[
\mathcal{M}_\tau^\vee \simeq \text{Hom}_{G(F)}(\tau \otimes \mathcal{M}_\tau, \tau) = \text{Hom}_{G(F)}(\mathcal{H}_\tau, \tau) \hookrightarrow \text{Hom}_{G(F)}(S, \tau)
\]

since \( \mathcal{H}_\tau \) is the completion of \( S \) w.r.t. the continuous semi-norm \( \| \alpha_\tau(\cdot) \|_{\mathcal{H}_\tau} \).
When $S = C_c^\infty(X^+)$, Bernstein (1988) showed

$$\text{Hom}_{G(F)}(S, \tau) \simeq \text{Hom}_{G(F)}(\pi, C^\infty(X^+)) = \mathcal{N}_\pi$$

where $\pi := \bar{\tau}^\infty$. Summing up, we obtain a relation

Spectral decomposition ($L^2$-theory) $\leftrightarrow$ distinction (smooth theory).

- $S = C_c^\infty(X^+)$ is easier to work with.
- Expectation: choice of $S$ is closely related to affine embeddings $X^+ \hookrightarrow X$.
- Different homogeneous spaces $X_1^+, X_2^+$ may have isometric $L^2$ induced from some equivariant $F : S_2 \simeq S_1$, which is difficult to see from $C_c^\infty$ test functions. Again, one may motivate by considering $L^2(\mathbb{R})$ and the usual Schwartz space.
Functional equations: $L^2$-aspect

Question: why should we expect functional equations?

1. Disintegrate $\mathcal{F} : L^2(X_2^+) \sim \rightarrow L^2(X_1^+)$ using the abstract Plancherel decomposition: get $\eta(\tau) : \mathcal{M}_\tau^{(2)} \rightarrow \mathcal{M}_\tau^{(1)}$ on multiplicity spaces. How to handle $\eta(\tau)$?

2. Suppose that $\mathcal{F}$ restricts to $S_2 \sim \rightarrow S_1$. By Gelfand-Kostyuchenko, $\mathcal{M}_\tau^{(i),\vee} \subset \text{Hom}_{G(F)}(S_i, \tau)$ and $\eta(\tau)^\vee$ is induced from a transport of structure via $\mathcal{F}$. How to describe the image of $\mathcal{M}_\tau^{(i),\vee}$?

3. It is relatively easier to identify $\mathcal{M}_\tau^{(i),\vee}$ as a subspace of $\mathcal{N}_\pi^{(i)}$ with $\pi = \bar{\tau}^\infty$. How to compare the relevant embeddings of $\pi$ into $C^\infty(X^+)$ and $S_i^\vee$?

4. My proposal: use zeta integrals + meromorphic continuation. This can be justified if for $X^+ = X_1^+, X_2^+$, $C_c^\infty(X^+) \rightarrow S$ induces

$$\text{Hom}_{G(F)}(\pi|\omega|^\lambda, S^\vee) \leftrightarrow \text{Hom}_{G(F)}(\pi|\omega|^\lambda, C_c^\infty(X^+)^\vee)$$

for $|\omega|^\lambda = \prod_i |\omega_i|^\lambda_i$ and $\lambda \in \Lambda_C$ in general position. Plus some technical assumptions...
Assuming the meromorphic continuation, etc. for zeta integrals, we deduce

\[ \mathcal{N}_\pi \leftrightarrow \mathcal{L}_\pi := \{ \text{nice meromorphic invariant } B_\lambda : \pi_\lambda \otimes S \rightarrow \mathbb{C} \} \]

with \( \pi_\lambda := \pi \otimes |\omega|\lambda \). It maps \( \varphi \) to \( B_\lambda (v \otimes \xi) = Z_\lambda (\xi, \varphi (v)) \).

Let \( \mathcal{K} \) be the field of meromorphic functions on \( \mathcal{T} := \{ |\omega|\lambda : \lambda \in \Lambda_{\mathbb{C}} \} \).

Obtain a \( \mathcal{K} \)-linear map \( \mathcal{N}_\pi \otimes \mathcal{K} \leftrightarrow \mathcal{L}_\pi \).

Now work with \( X_i^+ \leftrightarrow X_i \) for \( i = 1, 2 \). For ease of notations, assume \( \Lambda_1 = \Lambda_2 \), i.e. allow the same twists by characters \( |\omega|\lambda \).

**Definition**

The local functional equation for \( F : S_2 \simrightarrow S_1 \) and \( \pi \) holds if:

\[ \gamma (\pi) \leftrightarrow \gamma (\pi, \lambda) : \mathcal{N}_\pi^{(1)} \rightarrow \mathcal{N}_\pi^{(2)} \]

meromorphic in \( \lambda \).
In comparison of the $L^2$-aspect, one may infer (under some hypotheses as we have seen) that $\gamma(\pi, 0)$ restricts to $\eta(\tau)^\vee$ for almost all $\tau$ and $\pi := \tilde{\tau}\infty$. This is compatible with some known properties of $\gamma$-factors: unitarity along the critical line for tempered irreps, etc.

**Question**

How to obtain local functional equations in this sense?

In the non-archimedean case, one approach is to establish multiplicity-one of invariant bilinear forms $B_\lambda : \pi_\lambda \otimes S \to \mathbb{C}$ for general $\lambda$.

- This is closely related to the geometry of $\partial X$.
- It is possible to formulate a sufficient condition that implies Godement-Jacquet case, and probably some other prehomogeneous vector spaces.
Let \( F \) be a number field, \( \mathbb{A} = \mathbb{A}_F \).

- Same conditions on \( X^+ \hookrightarrow X \), assuming the Schwartz spaces \( S_v \) defined at each place \( v \).
- Assume that “basic functions” \( \xi_v \in S_v \) are chosen for almost all \( v \nmid \infty \) so that \( S = \bigotimes'_v S_v \) makes sense.
- Consider a continuous \( G(F) \)-invariant functional \( \vartheta : S \to \mathbb{C} \), eg.

\[
\vartheta(\xi) = \sum_{x \in X^+(F)} \text{ev}_\gamma(\xi)
\]

for appropriate \textit{evaluation maps} \( \text{ev}_\gamma \), with due care on the half-densities, etc.

- Frobenius reciprocity yields \( \xi \mapsto \vartheta_\xi \in C^\infty(G(F) \backslash G(\mathbb{A})) \).
It is natural to consider integrals

\[ Z_\lambda = \int_X \phi_\lambda \vartheta_\xi, \quad \Re(\lambda) \gg 0, \]

for \( \phi \): cusp form, \( \phi_\lambda = \phi|\omega|^\lambda \), \( X = G(F)\backslash G(\mathbb{A}) \) divided by some suitable central subgroup of \( G(F_\infty) \). Some issues:

- Convergence for \( \Re(\lambda) \gg 0 \) (look for sufficient conditions).
- Try to prove meromorphic continuation (hard).
- In principle, global functional equations should come from Poisson formulas relating Schwartz spaces of different spaces, as in the local case. Idealistically: \( \mathcal{F} : S_2 \overset{\sim}{\rightarrow} S_1 \), \( \vartheta^{(1)}(\mathcal{F}\xi) = \vartheta^{(2)}(\xi) \) (hard).
- One can relate \( Z_\lambda \) to period integrals in its range of convergence; when periods factorize, \( Z_\lambda \) also factorizes into local zeta integrals treated before.
Warning:
All statements above are subject to revision.

Work in progress...